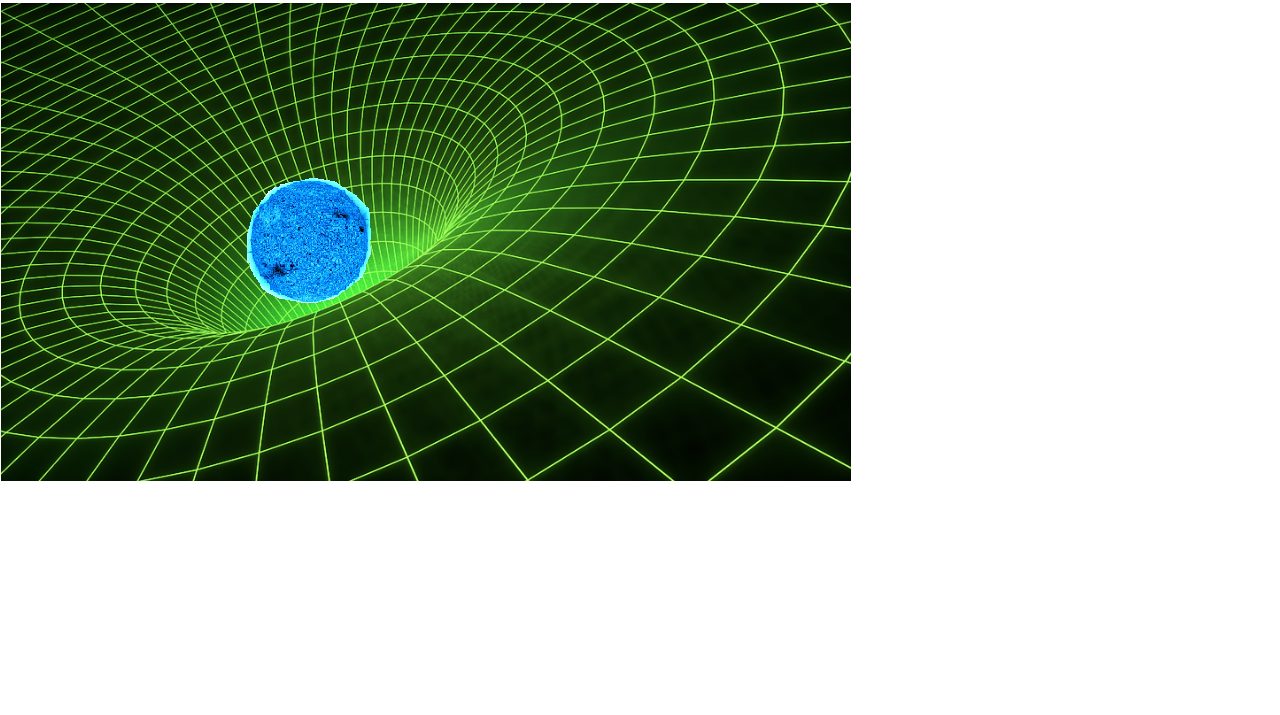
**Stars Interior**

I’ll just repeat myself here to make the interior star discussion self-contained. Now let’s explore another special case – stars. More generally, we’ll simply look at spherically symmetric arrangements, but these always describe stars.



Again we’ll start with Einstein’s equations:



Since stars are spherically symmetric, we will assume spherical symmetry in our metric ansatz. Additionally, we’ll assume that in our reference frame, the metric is independent of time, and also possesses time-reversal symmetry so that the star basically isn’t rotating. Let 0 go in the time-like direction, r go in the radial direction, θ in the theta directions, and φ in the phi direction. In that case we can write our metric as:



Note that t, r, θ, φ are coordinates in the time-like and radial-like, and theta-like, phi-like directions. But the former two are not identical to physical time and physical radial distance of course. What is the LHS of the Einstein equation? Let’s work it out. Our metric tensor is:



where T = T(r), R = R(r), Θ = r2, Φ = r2sin2θ, and e00, e11, e22, e33 are just place holders indicating the position of the elements in the matrix. Proceeding,



and



and so we should have for Γ,



The Ricci tensor is:



Now taking the derivative and trace at the same time, to form the first term in the Ricci tensor we have:



Next,



Next,



Next, and finally,



And now adding these together…we have:



Grouping together,



In particular, we’ve got:



The next term is:



and the next,



and the R33 guy,



The off-diagonal elements are zero, as can see…



and,



So so far, our metric is:



and our curvature tensor is so far:



and let’s get the curvature scalar:



So we have:



Now we must fill these expressions into the Einstein equation.



**Interior solution**

Now let’s solve for the metric in the interior of the star. We already have Rαβ and R, just like we had for the exterior star solution – no change there. What’s different is the stress-energy tensor. We’ll model the interior of the star as a perfect fluid so we have:



(P is presumed to depend on r) What is 2? Well, the star is stationary, by assumption, so the spatial components of the velocity are zero. But the magnitude of the four velocity is -c2, so the temporal component is not. We can work out the temporal component therefore.



So we can write:



So let’s put this all together.



Let’s look at the time component. We have:



We’ll take this moment to remind ourselves that ε0 is the rest energy density, and so does in principle depend on r. This is a pretty nasty equation. Well, actually it’s a Bernoulli equation too. So let’s proceed.



and so,



It’s customary to define the mass function:



where ρ(r) is the mass density. And in terms of this, we can write:



I feel like if ε0 = 0, then we should get R = 1. So then,



Now let’s look at the radial equation.



Now we have to solve this equation.



So we’d fill in R,



And we’ll note P can depend on r too, so let’s write this as:



Integrating this is a little difficult with respect to finding the integration constant. So I guess I’ll leave it here. I think the θ and φ components of the Einstein equations don’t add anything new, and in any event, we had only two independent variables T and R so only two equations are really independent. But we do have one more equation which governs the dynamics of the stress-energy tensor in response to the gravitational field:





where we’ll recall,



So we need to evaluate the covariant derivative. Well, first, we need:



Note the eαβ guys are just place holders for the positions of the elements; they’re not basis vectors. Now we take the derivative:



Now we found up above at the top of the file that the Christoffel-symbol was as follows:



where Θ = r2 and Φ = r2sin2θ. So, this simplifies to:



So let’s get the next term in the covariant derivative.



And the third term in the covariant derivative,



Now gotta put it all together:



Grouping things together:



Yikes. Now simplifying stuff:



Oh yeah, we want to say:



So this is our equation:



So these are our three equations. I’ll replace ε0(r) with ρ(r)c2.



These are the Tollman-Oppenheimer-Volkoff equations for the metric components. All one needs to complete them is an equation of state, or knowledge of ε(n,s).

**Interpretation of the mass M(r)**

We defined the mass function as:



where ρ(r) is ε0(r)/c2, the rest energy density/c2. But, if we were really trying to calculate the mass of the star, then we’d want to use the physical coordinate differentials. Recall from the Tensors file in the Appendix that the differential volume element is:



(determinant and product runs over spatial coordinates) which works out to:



Integrating over θ and φ, we’d get:



So the actual mass would be:



In the weak field limit, say where M(r) is small, then we have:



So we see that the actual mass differs from the regular mass by the additional gravitational potential energy. This is consistent with the fact that the actual mass is the physical rest energy, while the regular mass that we defined is just a mathematical construction that apparently is identical with the classical mass.

**Special case of constant density star**

So to make progress we need ε(n,s), or an equation of state. A simplification tantamount to this is to suppose we have a star of constant density: ρ(r) = ρ – not a very realistic case, but the easiest to work with. Our equations are:



and filling in our constant density, the M(r) comes to:



Then the radial function becomes,



Now our last two equations are:



We can plug the first into the last,



Rearranging stuff,



Let’s say the radius of the star is rs. Then we should have P = 0 here, at r = rs. And so,



Given that, we can write our solution as:



And we can solve for the pressure now.



Solving for the pressure, then:



Could write a little nicer, or at least more condensed:



Let’s introduce the mass of the star,



In terms of this, we can say:



So this becomes,



What is pressure at the center?



This will blow up when,



We can put this in terms of M alone, using our equation above: M = (4π/3)ρrs3. We find,



Simplifying a bit more, we see the mass of the star when Pc = ∞, i.e., the maximum star mass, is:



For nuclear matter densities, this equates to about 7.5Msolar. Past this point, the star would collapse into a black hole. Do we want to solve for T(r)?



Not really.